Points on Hemispheres

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Abstract

We will show that for any $n \geq N$ points on the N-dimensional sphere S^N there is a closed hemisphere which contains at least $\lfloor \frac{n+N+1}{2} \rfloor$ of these points. This bound is sharp and we will calculate the amount of sets which realize this value.

If we change to open hemispheres things will be easier. For any n points on the sphere there is an open hemisphere which contains at least $\lfloor \frac{n+1}{2} \rfloor$ of these points, independent of the dimension. This bound is sharp.

1 Introduction

In problem 451342 of the German Mathematical Olympiad one has to calculate the maximum of the minimal pairwise distance of five points on a (two-dimensional) sphere. For proving the result it was helpful to use the property that there is a (closed) hemisphere containing at least four of these points.

We now want to generalize this property to all numbers of points and all dimensions. The *N-dimensional sphere* S^N is the set of points in $\mathbb{R}^{N+1}$ with (Euclidean) distance 1 from the origin. Any (hyper-)plane through the origin splits the sphere into two *hemispheres*. The closed hemispheres contain the intersection set, which is called *great circle*, while the open hemispheres do not contain the great circle.

Any hemisphere has a unique *pole* p. Then the points x of the great circle, the closed and open hemisphere can be characterized by $\langle x, p\rangle = 0$, $\langle x, p\rangle \geq 0$, and $\langle x, p\rangle > 0$, respectively.

2 The lower bound

Lemma 2.1. *For any $n < N$ points on the N-dimensional sphere S^N, there is a closed hemisphere which contains all of these points.*

Proof. For $n < N$ points on the N-dimensional sphere there is always a great circle containing all these points. $\square$

Lemma 2.2. *For any n points on the N-dimensional sphere S^N, $n \geq N$, there is a closed hemisphere which contains at least $\lfloor \frac{n+N+1}{2} \rfloor$ of these points.*

Proof. Choose any N of these points. Then there is a plane through these points and the origin. So there is a great circle containing these N points. The remaining $n - N$ points distribute to the two hemispheres, hence the is a hemisphere containing at least $\lfloor \frac{n-N+1}{2} \rfloor$ of them. Together with the initial N points there are at least $\lfloor \frac{n+N+1}{2} \rfloor$ on this hemisphere. $\square$

3 The sharpness of the bound

Now we will check that the bound from the previous section is sharp.

Take any great circle and move it continuously until it contains N points. Then the number of points in any of the two hemispheres is non-decreasing. So no hemisphere contains more than $\lfloor \frac{n+N+1}{2} \rfloor$ points if and only if all the hemispheres whose border contains N points have this property.

Definition 3.1. A set of n points on the N-dimensional sphere S^N is called *equator-balanced set*, if any great circle through N of these points cuts the remaining points into halves. (If the number is odd then the two sets differ by one.)

Hence we have to show the existence of such sets.

Lemma 3.2. *For any $n > N$ there is an equator-balanced set of n points on the N-sphere.*

Proof. One can specify such a set. Let $y_i = (-1)^i \cdot (1, i, i^2, \ldots, i^N)$, $i = 1, \ldots, n$ be n vectors in $\mathbb{R}^{N+1}$, and x_i the corresponding normal vectors on the S^N.

Fix any N of these points. In which hemisphere the point x_i lies is determined by the determinant formed by x_i and the N points. Apart from some factors this is a Vandermonde determinant.

Replacing x_i by the next remaining point x_{i+j} the signum of the factor changes by $(-1)^j$ and in the Vandermonde determinant $j - 1$ factors change their sign. So the determinant has the opposite sign.

Hence the remaining points lie alternating in the two hemispheres, and the set is equator-balanced. $\square$

4 The density of equator-balanced sets

Now we can ask with which probability a set is equator-balanced.

Definition 4.1. By $p(N, n)$ we denote the probability that a set of n independent uniformly distributed points on the N-sphere is equator-balanced.

Lemma 4.2. *$p(n, N)$ exists and is always positive.*

Proof. In the configuration space of n-sets on the N-sphere let $M(n, N, k)$ be the set of all sets where any closed hemisphere contains at most k points. This is equivalent to: any open hemisphere contains at least $n - k$ points. But this is

an open condition, i.e. if a configuration is slightly changed then it also fulfills this condition. Hence $M(n, N, k)$ is an open set.

Since the equator-balanced sets constitute the set $M(n, N, k)$ with $k = \lfloor \frac{n+N+1}{2} \rfloor$ and by lemma 3.2 this set is non-empty, it is measurable with positive measurable. Hence $p(n, N)$ exists and is always positive. □

In some special cases $p(N, n)$ can be calculated. Obviously, if $n \leq N + 1$ then all sets are equator-balanced. So we will restrict to the case $n \geq N + 2$.

Lemma 4.3. $p(N, N + 2) = 2^{-(N+1)}$.

Proof. A set of $N + 2$ points on the N-sphere is not equator-balanced if and only if all the points lie in a common hemisphere.

Fix $N + 1$ of the points. They constitute a spherical $(N + 1)$-simplex. The only possibility for the last point not lying in a common hemisphere with the other points is to lie in the $(N + 1)$-simplex antipodal to the simplex.

Hence the probability $p(N, N + 2)$ is the expectation value of the volume of a random spherical $(N+1)$-simplex relative to the volume of the N-sphere. For independent random points $x_1, \ldots, x_{N+1}$ we have

$$E(V(x_1, \ldots, x_{N+1})) = E(V(\pm x_1, \ldots, \pm x_{N+1})) \tag{4.4}$$

for any combination of plus and minus signs. But all these 2^{N+1} simplices assembled together always gives the whole N-sphere, so the sum of these (equal) 2^{N+1} expectation values is 1, thus $p(N, N+2) = E(V(x_1, \ldots, x_{N+1})) = 2^{-(N+1)}$. □

Lemma 4.5. $p(1, 1 + 2k) = 4^{-k}$.

Proof. Any of the $1 + 2k$ can be replaced by its antipodal point. Consider all 2^{1+2k} combinations where each point can be replaced by its antipodal point. For all these combinations the probability to be equator-balanced is the same. We will show, that independently from the original configuration there are exactly 2 equator-balanced configurations among them. So the probability is $2/2^{1+2k} = 4^{-k}$.

Fix any diameter of the circle and count the points in one semicircle. Now rotate the diameter continuously half around and notice the change of the points in the semicircle. This numbers give a sequence of $2 + 2k$ numbers, where successive numbers differ by 1 and the first and last number add up to $1 + 2k$. The other way round any such sequence gives a configuration. The configuration is equator-balanced if and only if the sequence contains only the numbers k and $k + 1$. There are exactly two such sequences: $k, k + 1, \ldots, k + 1$ and $k + 1, k, \ldots, k$. □

In the preceeding examples $n - N$ was even, i.e. we coped with *real* equator-balanced sets. For the next interesting examples of this kind a stochastical simulation gave the results in table 1.

If $n - N$ is odd, then the amount of points in the two hemispheres bounded by a great circle trough N points differ by one. Hence there are much more possibilities for such configurations and the probability is much higher. But only in the case $N = 1$ we were able to determine the exact probability.

N	n	trials	success	$1/p(N,n)$	precision
2	6	287951134242	1708252518	168.5647	0.012
2	8	293892632084	23669718	12416.39	7.65
3	7	115638779856	42369783	2729.27	1.25
2	5	889631743	277996246	3.20015	0.00057
2	7	3558495944	254538093	13.98021	0.0026
2	9	11535004949	127320713	90.598	0.024

Table 1: Stochastic simulation for $p(N, n)$. Here the precision is the estimated 3σ-value for $1/p(N, n)$.

Lemma 4.6. $p(1, 2 + 2k) = 2^{-k}$.

Proof. As in the previous proof the calculation can be reduced to counting sequences. We have now sequences of length $3+2k$, where first and last number add up to $2+2k$. A configuration is equator-balanced if and only if the sequence contains only the numbers k, $k + 1$ and $k + 2$.

Hence any second number in the sequence is $k+1$, and the remaining numbers can be choosen arbitrarily k or $k + 2$ (except the last number). We have two possibilities to choose the "$k+1$-numbers" and again 2^{k+1} possibilities to choose the remaining numbers.

So we have 2^{k+2} such sequences, and the probability is $2^{k+2}/2^{2+2k} = 2^{-k}$. □

5 Open hemispheres

For open hemispheres the problem is much easier, because it is possible to "hide" points by pairs of antipodal points.

Lemma 5.1. *For any finite set of points on a sphere there is a great circle that does not contain any of these points.*

Proof. The poles of a great circle containing a given point lie on a great circle having the given point as a pole. Hence the poles of all great circles containing at least one of the points lie on a union of a finite number of great circles. So there are poles left for that the corresponding great circle does not contain any of these points. □

Lemma 5.2. *For any n points on the sphere, there is an open hemisphere which contains at least $\lfloor \frac{n+1}{2} \rfloor$ of these points. This bound is best possible.*

Proof. By Lemma 5.1 there is a great circle containing none of these points. So one of the corresponding hemispheres contains at least $\lfloor \frac{n+1}{2} \rfloor$ of these points.

If the points lie pairwise antipodal (and one additional point if n is odd), then any open hemisphere contains at most one point of any pair. Hence in this case any open hemisphere contains at most $\lfloor \frac{n+1}{2} \rfloor$ of these points. □